

# $N = 1$ Neveu-Schwarz vertex operator superalgebras over Grassmann algebras and with odd formal variables

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## Abstract

The notions of  $N = 1$  Neveu-Schwarz vertex operator superalgebra over a Grassmann algebra and with odd formal variables and of  $N = 1$  Neveu-Schwarz vertex operator superalgebra over a Grassmann algebra and without odd formal variables are introduced, and we show that the respective categories of such objects are isomorphic. The weak supercommutativity and weak associativity properties for an  $N = 1$  Neveu-Schwarz vertex operator superalgebra with odd formal variables are established, and we show that in the presence of the other axioms, weak supercommutativity and weak associativity are equivalent to the Jacobi identity. In addition, we prove the supercommutativity and associativity properties for an  $N = 1$  Neveu-Schwarz vertex operator superalgebra with odd formal variables and show that in the presence of the other axioms, supercommutativity and associativity are equivalent to the Jacobi identity.

## 1 Introduction

In this paper, we introduce the notion of  $N = 1$  *Neveu-Schwarz vertex operator superalgebra over a Grassmann algebra and with odd formal variables* and study the consequences of this notion. We also introduce the notion of  $N = 1$  *Neveu-Schwarz vertex operator superalgebra over a Grassmann algebra and without odd formal variables* and show that these two notions are equivalent in that the corresponding categories of such objects are isomorphic. Though many more or less equivalent notions of vertex operator superalgebra have been formulated (cf. [T], [G], [FFR], [DL], and [KW]), extending the notion

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of vertex operator algebra [Bo], [FLM], [FHL], none of these notions give the most natural setting for the corresponding supergeometry of  $N = 1$  superconformal field theory (cf. [Fd], [D]). Some lack a representation of the full  $N = 1$  Neveu-Schwarz algebra rather than just the Virasoro algebra and none are defined over a general Grassmann algebra. However, if one extends, for example, the notion of “ $N = 1$  NS-type SVOA” as given in [KW] to be a module over a general Grassmann algebra instead of just a vector space over  $\mathbb{C}$ , then one does obtain an equivalent notion to ours. Since the notions of  $N = 1$  Neveu-Schwarz vertex operator superalgebra over a Grassmann algebra and with and without odd formal variables are equivalent, one might very well ask why we should want to complicate matters by adding the odd formal variables. Below we give some motivation for including these odd components as well as motivation for working over a Grassmann algebra and requiring a representation of the  $N = 1$  Neveu-Schwarz algebra.

In [Ba1], we give a rigorous foundation to the correspondence between the geometric and algebraic aspects of genus-zero holomorphic  $N = 1$  superconformal field theory (cf. [Fd]), following the work of Huang [H2], [H1] by, introducing the notion of  $N = 1$  supergeometric vertex operator superalgebra and proving that the category of such objects is isomorphic to the category of  $N = 1$  Neveu-Schwarz vertex operator superalgebras over a Grassmann algebra and with (or without) odd formal variables. The supergeometry of  $N = 1$  superconformal field theory is defined over a Grassmann algebra (cf. [D], [R]), which is why it is more natural to work over such an algebra rather than just  $\mathbb{C}$ . This supergeometry and the notion of  $N = 1$  supergeometric vertex operator superalgebra involve the moduli space of  $N = 1$  superspheres with tubes (corresponding to the interaction of incoming and outgoing superstrings) which naturally has even and odd variables in the underlying Grassmann algebra.

Recall that in a vertex operator algebra, the Virasoro element  $L(-1)$  is related to the differential operator  $\frac{\partial}{\partial z}$  via the  $L(-1)$ -derivative property (see [FHL]). This correspondence can be thought of as a correspondence between the algebraic setting and the geometric setting of genus-zero holomorphic conformal field theory as developed in [H1] and [H2] following [BPZ], [FS], [V], [S]. In the geometry of conformal field theory, local coordinates are conformal in that they transform the differential operator  $\frac{\partial}{\partial z}$  homogeneously of degree one for  $z$  a complex variable. In  $N = 1$  superconformal field theory, the natural setting is  $N = 1$  supergeometry, and local coordinates which transform a certain superdifferential operator  $D$  homogeneously of degree one

where  $D$  satisfies  $D^2 = \frac{\partial}{\partial z}$ . Such an operator is given by  $D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z}$  where  $z$  is an even variable over a Grassmann algebra and  $\theta$  is an odd variable over a Grassmann algebra (cf. [Fd], [Ba1], [Ba2]). In addition, in [Ba1], we show that the Lie superalgebra of infinitesimal local coordinate transformations for an  $N = 1$  supersphere with tubes is isomorphic to the  $N = 1$  Neveu-Schwarz algebra [NS] with central charge zero.

Thus in considering what should be the corresponding superalgebraic setting for  $N = 1$  supergeometric vertex operator superalgebra, we naturally want to have a representation of the  $N = 1$  Neveu-Schwarz algebra in which the element  $G(-\frac{1}{2})$  has the supercommutator  $\frac{1}{2}[G(-\frac{1}{2}), G(-\frac{1}{2})] = L(-1)$  (which corresponds to  $G(-\frac{1}{2})^2 = L(-1)$  in the universal enveloping algebra of the Neveu-Schwarz algebra). Furthermore, this operator  $G(-\frac{1}{2})$  should be related to the superdifferential operator  $D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z}$  consistent with the relationship between the  $L(-1)$  operator and  $\frac{\partial}{\partial z}$ . In our notion of  $N = 1$  Neveu-Schwarz vertex operator superalgebra over a Grassmann algebra and with odd formal variables, we have an  $N = 1$  Neveu-Schwarz algebra element giving rise to a representation of the  $N = 1$  Neveu-Schwarz algebra with a corresponding  $G(-\frac{1}{2})$ -derivative property (property (22) in Definition 4.1 below) giving the explicit connection between the endomorphism  $G(-\frac{1}{2})$  and the superderivation  $D$ . The  $L(-1)$ -derivative property then follows from this more fundamental property.

Furthermore, the correlation functions of  $N = 1$  superconformal field theory (cf. [Fd]) are superanalytic superfunctions in one even and one odd variable and, in the genus-zero holomorphic case, correspond to the correlation functions arising from an  $N = 1$  Neveu-Schwarz vertex operator superalgebra over a Grassmann algebra and with odd formal variables as studied in Section 9 of this paper, i.e., our inclusion of the odd formal variables gives the entire correlation function explicitly.

Thus the fact that we are working over a general Grassmann algebra and the presence of a representation of the  $N = 1$  Neveu-Schwarz algebra, the  $G(-\frac{1}{2})$ -derivative property, and the odd formal variable component of the vertex operators in our definition of  $N = 1$  Neveu-Schwarz vertex operator algebra over a Grassmann algebra and with odd formal variables give a more natural correspondence to the supergeometry of  $N = 1$  superconformal field theory.

For the remainder of this paper we will often use the term “vertex operator superalgebra” to mean “ $N = 1$  Neveu-Schwarz vertex operator superalgebra over a Grassmann algebra” unless otherwise noted.

That the notions of vertex operator superalgebra with and without odd formal variables are equivalent is due to the fact that all of the information for the odd variable components of the vertex operators is contained in the representation of the Neveu-Schwarz algebra. In fact, given a vertex operator superalgebra without odd formal variables there is only one way to form the odd variable component in order to obtain a vertex operator superalgebra with odd formal variables and that is by employing the  $G(-\frac{1}{2})$  representative element (see consequence (32) of Definition 4.1 below).

After introducing the notions of vertex operator superalgebra with and without odd formal variables and proving their equivalence, we formulate the properties of weak supercommutativity and weak associativity for a vertex operator superalgebra with odd formal variables. Following [DL], [L1] and [L2], we show that in the presence of the other axioms for a vertex operator superalgebra with odd formal variables, weak supercommutativity and weak associativity are equivalent to the Jacobi identity. Then following [FLM] and [FHL], we look at expansions of rational superfunctions and formulate the properties of rationality of products and iterates, supercommutativity and associativity for a vertex operator superalgebra with odd formal variables. Finally, following [FHL], we show that in the presence of the other axioms, rationality of products and iterates, supercommutativity and associativity are equivalent to the Jacobi identity. These properties are used in the proof that the category of  $N = 1$  supergeometric vertex operator superalgebras is isomorphic to the category of vertex operator superalgebras with odd formal variables given in [Ba1] and announced in [Ba2].

The results in this paper follow the theory of vertex operator algebras as developed in [Bo], [FLM], [FHL], [DL] and [L2] and are contained in the author's Ph.D. dissertation [Ba1].

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## 2 Grassmann algebras and the $N = 1$ Neveu-Schwarz algebra

Let  $\mathbb{Z}_2$  denote the integers modulo two. For a  $\mathbb{Z}_2$ -graded vector space  $V = V^0 \oplus V^1$ , define the *sign function*  $\eta$  on the homogeneous subspaces of  $V$  by  $\eta(v) = i$  for  $v \in V^i$ ,  $i = 0, 1$ . If  $\eta(v) = 0$ , we say that  $v$  is *even*, and if  $\eta(v) = 1$ , we say that  $v$  is *odd*. (Note that with this definition, the sign function is double-valued at zero. However, in practice this is never an issue, as can be seen in the definitions below.)

A *superalgebra* is an (associative) algebra  $A$  (with identity  $1 \in A$ ), such that

- (i)  $A$  is a  $\mathbb{Z}_2$ -graded algebra
- (ii)  $ab = (-1)^{\eta(a)\eta(b)}ba$  for  $a, b$  homogeneous in  $A$ .

The exterior algebra over a vector space  $U$ , denoted  $\Lambda(U)$ , has the structure of a superalgebra. Fix  $U_L$  to be an  $L$ -dimensional vector space over  $\mathbb{C}$  for  $L \in \mathbb{N}$  such that  $U_L \subset U_{L+1}$ . We denote  $\Lambda(U_L)$  by  $\Lambda_L$  and call this the *Grassmann algebra on  $L$  generators*. Note that  $\Lambda_L \subset \Lambda_{L+1}$ , and taking the direct limit as  $L \rightarrow \infty$ , we have the *infinite Grassmann algebra* denoted by  $\Lambda_\infty$ . We use the notation  $\Lambda_*$  to denote a Grassmann algebra, finite or infinite. Since this paper is motivated by the geometry of  $N = 1$  superconformal field theory in which one considers superanalytic structures, for the purposes of this paper we have defined  $\Lambda_*$  over  $\mathbb{C}$ . However, we could just as well have formulated the results that follow for Grassmann algebras over any field of characteristic zero.

A  $\mathbb{Z}_2$ -graded vector space  $\mathfrak{g}$  is said to be a *Lie superalgebra* if it has a bilinear operation  $[\cdot, \cdot]$  such that for  $u, v$  homogeneous in  $\mathfrak{g}$ ,

- (i)  $[u, v] \in \mathfrak{g}^{(\eta(u)+\eta(v)) \bmod 2}$
- (ii)  $[u, v] = -(-1)^{\eta(u)\eta(v)}[v, u]$  (skew-symmetry)
- (iii)  $(-1)^{\eta(u)\eta(w)}[[u, v], w] + (-1)^{\eta(v)\eta(u)}[[v, w], u] + (-1)^{\eta(w)\eta(v)}[[w, u], v] = 0.$  (Jacobi identity)

The Virasoro algebra is the Lie algebra with central charge  $d$ , basis consisting of the central element  $d$  and  $L_n$ , for  $n \in \mathbb{Z}$ , and commutation relations

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0} d, \quad (1)$$

for  $m, n \in \mathbb{Z}$ . The  $N = 1$  Neveu-Schwarz Lie superalgebra is a super-extension of the Virasoro algebra by the odd elements  $G_{n+\frac{1}{2}}$ , for  $n \in \mathbb{Z}$ . That is, the  $N = 1$  Neveu-Schwarz algebra has a basis consisting of the central element  $d$ ,  $L_n$  and  $G_{n+\frac{1}{2}}$ , for  $n \in \mathbb{Z}$ , with supercommutation relations

$$\left[ G_{m+\frac{1}{2}}, L_n \right] = \left( m - \frac{n-1}{2} \right) G_{m+n+\frac{1}{2}} \quad (2)$$

$$\left[ G_{m+\frac{1}{2}}, G_{n-\frac{1}{2}} \right] = 2L_{m+n} + \frac{1}{3}(m^2 + m)\delta_{m+n,0} d \quad (3)$$

in addition to (1).

### 3 Delta functions with odd formal variables

Let  $x_1$  and  $x_2$  be formal variables which commute with each other and with  $\Lambda_*$ . We call such a variable an *even formal variable*. Let  $\varphi_1$  and  $\varphi_2$  be formal variables which commute with  $x_1, x_2$  and  $\Lambda_*^0$  and anticommute with each other and  $\Lambda_*^1$ . We call such variables *odd formal variables*. Note that the square of any odd formal variable is zero. Thus for any formal Laurent series  $f(x) \in \Lambda_*[[x, x^{-1}]]$ , we can define

$$f(x + \varphi_1 \varphi_2) = f(x) + \varphi_1 \varphi_2 f'(x) \in \Lambda_*[[x, x^{-1}]][\varphi_1][\varphi_2]. \quad (4)$$

In order to formulate the notion of vertex operator superalgebra with odd formal variables, we would like to use the *formal  $\delta$ -function at  $x = 1$*  given by

$$\delta(x) = \sum_{n \in \mathbb{Z}} x^n,$$

which is a fundamental ingredient in the formal calculus underlying the theory of vertex operator algebras. However, we will want to extend the  $\delta$ -function to be defined for certain expressions involving both even and odd formal variables using (4).

Let  $x, x_0, x_1$ , and  $x_2$  be even formal variables. Following the treatment of formal calculus for even formal variables given in [FHL], we have

$$f(x)\delta(x) = f(1)\delta(x) \quad \text{for } f(x) \in \mathbb{C}[x, x^{-1}]. \quad (5)$$

Let  $V$  be a vector space over  $\mathbb{C}$ . For  $X(x_1, x_2) \in (\text{End } V)[[x_1, x_1^{-1}, x_2, x_2^{-1}]]$  such that

$$\lim_{x_1 \rightarrow x_2} X(x_1, x_2) \text{ exists, i.e., } X(x_1, x_2)|_{x_1=x_2} \text{ exists}$$

(that is, when  $X(x_1, x_2)$  is applied to any element of  $V$ , setting the variables equal leads to only finite sums in  $V$ ), we have

$$X(x_1, x_2)\delta\left(\frac{x_1}{x_2}\right) = X(x_2, x_2)\delta\left(\frac{x_1}{x_2}\right). \quad (6)$$

Let  $\mathbb{N}$  denote the natural numbers. For the three-variable generating function

$$\delta\left(\frac{x_1 - x_2}{x_0}\right) = \sum_{n \in \mathbb{Z}} \frac{(x_1 - x_2)^n}{x_0^n} = \sum_{n \in \mathbb{Z}, m \in \mathbb{N}} (-1)^m \binom{n}{m} x_0^{-n} x_1^{n-m} x_2^m,$$

we have

$$x_1^{-1} \delta\left(\frac{x_2 + x_0}{x_1}\right) = x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) \quad (7)$$

and

$$x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) - x_0^{-1} \delta\left(\frac{x_2 - x_1}{-x_0}\right) = x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right), \quad (8)$$

where we use the convention that any expression such as  $(x_1 - x_2)^n$  for  $n \in \mathbb{Z}$ , is understood to be expanded in positive powers of  $x_2$ . We will continue to use this convention throughout the rest of this work.

Notice that in the spirit of the  $\delta$ -function multiplication principle (6), the expressions on both sides of (7) and the three terms occurring in (8) all correspond to the same formal substitution

$$x_0 = x_1 - x_2.$$

Let  $\varphi$ ,  $\varphi_1$ , and  $\varphi_2$  be odd formal variables. Extending the above results, we see that for any  $f(x, \varphi) \in \bigwedge_*[x, x^{-1}, \varphi]$ , the following is an immediate consequences of (5):

$$f(x, \varphi)\delta(x) = f(1, \varphi)\delta(x).$$

Let  $V$  be a  $\bigwedge_*$ -module. If  $X(x_1, \varphi_1, x_2, \varphi_2) \in (\text{End } V)[[x_1, x_1^{-1}, x_2, x_2^{-1}]][\varphi_1, \varphi_2]$  such that

$$\lim_{x_1 \rightarrow x_2} X(x_1, \varphi_1, x_2, \varphi_2) \text{ exists, i.e., } X(x_1, \varphi_1, x_2, \varphi_2)|_{x_1=x_2} \text{ exists,}$$

then we have the following immediate consequence of (6):

$$X(x_1, \varphi_1, x_2, \varphi_2)\delta\left(\frac{x_1}{x_2}\right) = X(x_2, \varphi_1, x_2, \varphi_2)\delta\left(\frac{x_1}{x_2}\right). \quad (9)$$

We have the following  $\delta$ -function of expressions involving three even variables and two odd variables

$$\begin{aligned}
\delta\left(\frac{x_1 - x_2 - \varphi_1\varphi_2}{x_0}\right) &= \sum_{n \in \mathbb{Z}} (x_1 - x_2 - \varphi_1\varphi_2)^n x_0^{-n} \\
&= \sum_{n \in \mathbb{Z}} \left( (x_1 - x_2)^n - n\varphi_1\varphi_2(x_1 - x_2)^{n-1} \right) x_0^{-n} \\
&= \delta\left(\frac{x_1 - x_2}{x_0}\right) - \varphi_1\varphi_2 x_0^{-1} \delta'\left(\frac{x_1 - x_2}{x_0}\right)
\end{aligned}$$

where

$$\delta'(x) = \frac{d}{dx} \delta(x) = \sum_{n \in \mathbb{Z}} n x^{n-1},$$

and we use the convention that a function of even and odd variables should be expanded about the even variables. Conceptually, however, we can also think of  $\delta\left(\frac{x_1 - x_2 - \varphi_1\varphi_2}{x_0}\right)$  as being a function in four even variables where  $\varphi_1\varphi_2$  is considered as another even formal variable  $x_3$  with the property that  $x_3^2 = 0$ .

Taking  $\frac{\partial}{\partial x_0}$  of both sides of (7), we obtain the following identity:

$$x_1^{-2} \delta'\left(\frac{x_2 + x_0}{x_1}\right) = -x_2^{-2} \delta'\left(\frac{x_1 - x_0}{x_2}\right). \quad (10)$$

Taking  $\frac{\partial}{\partial x_1}$  of both sides of (8), we obtain:

$$x_0^{-2} \delta'\left(\frac{x_1 - x_2}{x_0}\right) - x_0^{-2} \delta'\left(\frac{x_2 - x_1}{-x_0}\right) = x_2^{-2} \delta'\left(\frac{x_1 - x_0}{x_2}\right). \quad (11)$$

Thus from (7), (8), (10) and (11), we have

$$x_1^{-1} \delta\left(\frac{x_2 + x_0 + \varphi_1\varphi_2}{x_1}\right) = x_2^{-1} \delta\left(\frac{x_1 - x_0 - \varphi_1\varphi_2}{x_2}\right) \quad (12)$$

and

$$\begin{aligned}
x_0^{-1} \delta\left(\frac{x_1 - x_2 - \varphi_1\varphi_2}{x_0}\right) - x_0^{-1} \delta\left(\frac{x_2 - x_1 + \varphi_1\varphi_2}{-x_0}\right) = \\
x_2^{-1} \delta\left(\frac{x_1 - x_0 - \varphi_1\varphi_2}{x_2}\right).
\end{aligned} \quad (13)$$



Notice that in the spirit of the  $\delta$ -function multiplication principle (9), the expressions on both sides of (12) and the three terms occurring in (13) all correspond to the same formal substitution

$$x_0 = x_1 - x_2 - \varphi_1\varphi_2.$$

And thus, not surprisingly, for

$$X(x_0, \varphi_0, x_1, \varphi_1, x_2, \varphi_2) \in (\text{End } V)[[x_0, x_0^{-1}, x_1, x_1^{-1}, x_2, x_2^{-1}]][\varphi_1, \varphi_2],$$

a formal substitution corresponding to  $x_0 = x_1 - x_2 - \varphi_1\varphi_2$  can be made as long as the resulting expression is well defined, e.g., if

$$X(x_1, \varphi_1, x_2, \varphi_2) \in (\text{End } V)[[x_1, x_1^{-1}]]((x_2))[\varphi_1, \varphi_2],$$

then

$$\delta\left(\frac{x_2 + x_0 + \varphi_1\varphi_2}{x_1}\right)X(x_1, \varphi_1, x_2, \varphi_2) = \tag{14}$$

$$\delta\left(\frac{x_2 + x_0 + \varphi_1\varphi_2}{x_1}\right)X(x_2 + x_0 + \varphi_1\varphi_2, \varphi_1, x_2, \varphi_2).$$

The substitution  $x_0 = x_1 - x_2 - \varphi_1\varphi_2$  can be thought of as the even part of a superconformal shift of  $(x_1, \varphi_1)$  by  $(x_2, \varphi_2)$ . Formally, a power series  $f(x_1, \varphi_1) \in \bigwedge_*[[x_1]][\varphi_1]$  is superconformal in  $x_1$  and  $\varphi_1$  if and only if  $f$  satisfies  $D\tilde{x} = \tilde{\varphi}D\tilde{\varphi}$  for  $f(x_1, \varphi_1) = (\tilde{x}, \tilde{\varphi})$  and  $D = \frac{\partial}{\partial\varphi_1} + \varphi_1\frac{\partial}{\partial x_1}$  (see [Ba1]). (For a superanalytic function  $f$  in one even variable and one odd variable this condition is equivalent to requiring that  $f$  transform the superdifferential operator  $D$  homogeneously of degree one.) Thus  $f(x_1, \varphi_1) = (x_1 - x_2 - \varphi_1\varphi_2, \varphi_1 - \varphi_2)$  is formally superconformal in  $x_1$  and  $\varphi_1$  since

$$\begin{aligned} D\tilde{x} &= \left(\frac{\partial}{\partial\varphi_1} + \varphi_1\frac{\partial}{\partial x_1}\right)(x_1 - x_2 - \varphi_1\varphi_2) \\ &= -\varphi_2 + \varphi_1 \\ &= (\varphi_1 - \varphi_2)\left(\frac{\partial}{\partial\varphi_1} + \varphi_1\frac{\partial}{\partial x_1}\right)(\varphi_1 - \varphi_2) \\ &= \tilde{\varphi}D\tilde{\varphi}. \end{aligned}$$

## 4 The notion of vertex operator superalgebra over $\bigwedge_*$ and with odd formal variables

**Definition 4.1** A  $(N = 1 \text{ Neveu-Schwarz})$  vertex operator superalgebra over  $\bigwedge_*$  and with odd variables is a  $\frac{1}{2}\mathbb{Z}$ -graded (by weight)  $\bigwedge_*$ -module which is also  $\mathbb{Z}_2$ -graded (by sign)

$$V = \coprod_{n \in \frac{1}{2}\mathbb{Z}} V_{(n)} = \coprod_{n \in \frac{1}{2}\mathbb{Z}} V_{(n)}^0 \oplus \coprod_{n \in \frac{1}{2}\mathbb{Z}} V_{(n)}^1 = V^0 \oplus V^1 \quad (15)$$

such that

$$\dim V_{(n)} < \infty \quad \text{for } n \in \frac{1}{2}\mathbb{Z}, \quad (16)$$

$$V_{(n)} = 0 \quad \text{for } n \text{ sufficiently small}, \quad (17)$$

equipped with a linear map  $V \otimes V \longrightarrow V[[x, x^{-1}]][\varphi]$ , or equivalently,

$$\begin{aligned} V &\longrightarrow (\text{End } V)[[x, x^{-1}]][\varphi] \\ v &\mapsto Y(v, (x, \varphi)) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} + \varphi \sum_{n \in \mathbb{Z}} v_{n-\frac{1}{2}} x^{-n-1} \end{aligned}$$

where  $v_n \in (\text{End } V)^{\eta(v)}$  and  $v_{n-\frac{1}{2}} \in (\text{End } V)^{(\eta(v)+1) \bmod 2}$  for  $v$  of homogeneous sign in  $V$ ,  $x$  is an even formal variable, and  $\varphi$  is an odd formal variable, and where  $Y(v, (x, \varphi))$  denotes the *vertex operator associated with*  $v$ , and equipped also with two distinguished homogeneous vectors  $\mathbf{1} \in V_{(0)}^0$  (the *vacuum*) and  $\tau \in V_{(\frac{3}{2})}^1$  (the *Neveu-Schwarz element*). The following conditions are assumed for  $u, v \in V$ :

$$u_n v = 0 \quad \text{for } n \in \frac{1}{2}\mathbb{Z} \text{ sufficiently large}; \quad (18)$$

$$Y(\mathbf{1}, (x, \varphi)) = 1 \quad (1 \text{ on the right being the identity operator}); \quad (19)$$

the *creation property* holds:

$$Y(v, (x, \varphi))\mathbf{1} \in V[[x]][\varphi] \quad \text{and} \quad \lim_{(x, \varphi) \rightarrow 0} Y(v, (x, \varphi))\mathbf{1} = v;$$

the *Jacobi identity* holds:

$$x_0^{-1} \delta\left(\frac{x_1 - x_2 - \varphi_1 \varphi_2}{x_0}\right) Y(u, (x_1, \varphi_1)) Y(v, (x_2, \varphi_2))$$

$$\begin{aligned}
& -(-1)^{\eta(u)\eta(v)}x_0^{-1}\delta\left(\frac{x_2-x_1+\varphi_1\varphi_2}{-x_0}\right)Y(v,(x_2,\varphi_2))Y(u,(x_1,\varphi_1)) \\
& = x_2^{-1}\delta\left(\frac{x_1-x_0-\varphi_1\varphi_2}{x_2}\right)Y(Y(u,(x_0,\varphi_1-\varphi_2))v,(x_2,\varphi_2)),
\end{aligned}$$

for  $u, v$  of homogeneous sign in  $V$ ; the Neveu-Schwarz algebra relations hold:

$$\begin{aligned}
[L(m), L(n)] &= (m-n)L(m+n) + \frac{1}{12}(m^3-m)\delta_{m+n,0}(\text{rank } V), \\
\left[G(m+\frac{1}{2}), L(n)\right] &= (m-\frac{n-1}{2})G(m+n+\frac{1}{2}), \\
\left[G(m+\frac{1}{2}), G(n-\frac{1}{2})\right] &= 2L(m+n) + \frac{1}{3}(m^2+m)\delta_{m+n,0}(\text{rank } V),
\end{aligned}$$

for  $m, n \in \mathbb{Z}$ , where

$$G(n+\frac{1}{2}) = \tau_{n+1}, \quad \text{and} \quad 2L(n) = \tau_{n+\frac{1}{2}} \quad \text{for } n \in \mathbb{Z},$$

i.e.,

$$Y(\tau, (x, \varphi)) = \sum_{n \in \mathbb{Z}} G(n+\frac{1}{2})x^{-n-\frac{1}{2}-\frac{3}{2}} + 2\varphi \sum_{n \in \mathbb{Z}} L(n)x^{-n-2}, \quad (20)$$

and  $\text{rank } V \in \mathbb{C}$ ;

$$L(0)v = nv \quad \text{for } n \in \frac{1}{2}\mathbb{Z} \quad \text{and} \quad v \in V_{(n)}; \quad (21)$$

$$\left(\frac{\partial}{\partial \varphi} + \varphi \frac{\partial}{\partial x}\right)Y(v, (x, \varphi)) = Y(G(-\frac{1}{2})v, (x, \varphi)). \quad (22)$$

The vertex operator superalgebra just defined is denoted by

$$(V, Y(\cdot, (x, \varphi)), \mathbf{1}, \tau)$$

or for simplicity by  $V$ .

For such a vertex operator algebra  $V$  over  $\bigwedge_*$  for  $\bigwedge_* = \bigwedge_0 = \mathbb{C}$ , the  $\mathbb{Z}_2$ -grading is given by

$$V^0 = \coprod_{n \in \mathbb{Z}} V_{(n)} \quad \quad V^1 = \coprod_{n \in \mathbb{Z}+\frac{1}{2}} V_{(n)}.$$

Extending  $V$  to a vertex operator superalgebra over a general  $\Lambda_*$  by  $\Lambda_* \otimes V$  changes the  $\mathbb{Z}_2$ -grading via  $(\Lambda_* \otimes V)^0 = \Lambda_*^0 \otimes V^0 + \Lambda_*^1 \otimes V^1$  and  $(\Lambda_* \otimes V)^1 = \Lambda_*^1 \otimes V^0 + \Lambda_*^0 \otimes V^1$ .

Since  $\Lambda_L \subset \Lambda_\infty$  for  $L \in \mathbb{N}$ , any vertex operator superalgebra  $V$  over  $\Lambda_L$  can be extended to a vertex operator superalgebra over  $\Lambda_\infty$  by, for instance, defining the action of  $\Lambda_\infty \setminus \Lambda_L$  on  $V$  to be trivial, or by taking the  $\Lambda_\infty$ -module induced by the  $\Lambda_L$ -module  $V$ .

Extending the proofs for the analogous properties in the non-super case from [FHL], we have the following consequences of the definition of vertex operator superalgebra with odd formal variables.

$$L(n)\mathbf{1} = 0, \quad \text{and} \quad G(n + \frac{1}{2})\mathbf{1} = 0, \quad \text{for } n \geq -1; \quad (23)$$

$$G(-\frac{3}{2})\mathbf{1} = \tau; \quad (24)$$

$$L(0)\tau = \frac{3}{2}\tau; \quad (25)$$

$$\begin{aligned} Y(v, (x, \varphi))\mathbf{1} &= e^{xL(-1) + \varphi G(-\frac{1}{2})}v \\ &= e^{xL(-1)}v + \varphi G(-\frac{1}{2})e^{xL(-1)}v; \end{aligned} \quad (26)$$

there exists  $\omega = \frac{1}{2}G(-\frac{1}{2})\tau \in V_{(2)}$  such that

$$Y(\omega, (x, \varphi)) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2} - \frac{\varphi}{2} \sum_{n \in \mathbb{Z}} (n+1)G(n - \frac{1}{2})x^{-n-2}; \quad (27)$$

$$\text{wt } v_n = \text{wt } v - n - 1, \quad (28)$$

for  $n \in \frac{1}{2}\mathbb{Z}$  and for  $v \in V$  of homogeneous weight.

Applying the  $G(-\frac{1}{2})$ -derivative property (22) twice, we obtain the  $L(-1)$ -derivative property

$$\frac{\partial}{\partial x} Y(v, (x, \varphi)) = Y(L(-1)v, (x, \varphi)). \quad (29)$$

Using the other properties, we see that the  $N = 1$  Neveu-Schwarz algebra supercommutation relations are equivalent to:

$$Y(\tau, (x, \varphi))\tau = \frac{2}{3}(\text{rank } V)\mathbf{1}x^{-3} + 2\omega x^{-1} + \varphi(3\tau x^{-2} + 2L(-1)\tau x^{-1}) + y \quad (30)$$

where  $y \in V[[x]][\varphi]$ .

Taking  $\text{Res}_{x_0}$  of the Jacobi identity and using the  $\delta$ -function identity (13), we obtain the following supercommutator formula where  $\text{Res}_{x_0}$  of a power series in  $x_0$  is the coefficient of  $x_0^{-1}$ .

$$[Y(u, (x_1, \varphi_1)), Y(v, (x_2, \varphi_2))] = \quad (31)$$

$$\text{Res}_{x_0} x_2^{-1} \delta\left(\frac{x_1 - x_0 - \varphi_1 \varphi_2}{x_2}\right) Y(Y(u, (x_0, \varphi_1 - \varphi_2))v, (x_2, \varphi_2)).$$

From the  $G(-\frac{1}{2})$ -derivative property (22) and the supercommutator formula (31), we have

$$Y(v, (x, \varphi)) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} + \varphi \sum_{n \in \mathbb{Z}} [G(-\frac{1}{2}), v_n] x^{-n-1}, \quad (32)$$

i.e.,  $v_{n-\frac{1}{2}} = [G(-\frac{1}{2}), v_n]$ .

Taking  $\text{Res}_{x_0} \text{Res}_{x_1}$  of the Jacobi identity, we find that for  $u, v \in V$

$$[u_{-\frac{1}{2}}, Y(v, (x, \varphi))] = Y(u_{-\frac{1}{2}}v, (x, \varphi)) \quad (33)$$

and in particular,

$$[u_{-\frac{1}{2}}, v_n] = (u_{-\frac{1}{2}}v)_n \quad \text{for } n \in \frac{1}{2}\mathbb{Z}, \quad (34)$$

and

$$[u_{-\frac{1}{2}}, v_{-\frac{1}{2}}] = (u_{-\frac{1}{2}}v)_{-\frac{1}{2}}; \quad (35)$$

thus the operators  $u_{-\frac{1}{2}}$  form a Lie superalgebra.

From Taylor's Theorem for formal calculus (cf. [FHL]) and the  $L(-1)$ - and  $G(-\frac{1}{2})$ -derivative properties (29) and (22), we have

$$Y(e^{x_0 L(-1)}v, (x, \varphi)) = e^{x_0 \frac{\partial}{\partial x}} Y(v, (x, \varphi)) \quad (36)$$

$$= Y(v, (x_0 + x, \varphi)) \quad (37)$$

$$Y(e^{\varphi_0 G(-\frac{1}{2})}v, (x, \varphi)) = e^{\varphi_0 (\frac{\partial}{\partial \varphi} + \varphi \frac{\partial}{\partial x})} Y(v, (x, \varphi)) \quad (38)$$

$$= Y(v, (x + \varphi_0 \varphi, \varphi_0 + \varphi)), \quad (39)$$

and thus

$$Y(e^{x_0 L(-1) + \varphi_0 G(-\frac{1}{2})}v, (x, \varphi)) = e^{x_0 \frac{\partial}{\partial x} + \varphi_0 (\frac{\partial}{\partial \varphi} + \varphi \frac{\partial}{\partial x})} Y(v, (x, \varphi)) \quad (40)$$

$$= Y(v, (x_0 + x + \varphi_0 \varphi, \varphi_0 + \varphi)). \quad (41)$$

Taking  $\text{Res}_{x_1}$  of the supercommutator formula (31) with  $u = \tau$  or with  $u = \omega = \frac{1}{2}G(-\frac{1}{2})\tau$ , we have

$$\left[ L(-1), Y(v, (x, \varphi)) \right] = Y(L(-1)v, (x, \varphi)) \quad (42)$$

$$\left[ G(-\frac{1}{2}), Y(v, (x, \varphi)) \right] = Y(G(-\frac{1}{2})v, (x, \varphi)) - 2\varphi Y(L(-1)v, (x, \varphi)) \quad (43)$$

$$\begin{aligned} \left[ L(0), Y(v, (x, \varphi)) \right] &= Y(L(0)v, (x, \varphi)) + \frac{\varphi}{2} Y(G(-\frac{1}{2})v, (x, \varphi)) \\ &\quad + xY(L(-1)v, (x, \varphi)) \end{aligned} \quad (44)$$

$$\begin{aligned} \left[ G(\frac{1}{2}), Y(v, (x, \varphi)) \right] &= Y(G(\frac{1}{2})v, (x, \varphi)) - 2\varphi Y(L(0)v, (x, \varphi)) \\ &\quad + xY(G(-\frac{1}{2})v, (x, \varphi)) - 2x\varphi Y(L(-1)v, (x, \varphi)) \end{aligned} \quad (45)$$

$$\begin{aligned} \left[ L(1), Y(v, (x, \varphi)) \right] &= Y(L(1)v, (x, \varphi)) + \varphi Y(G(\frac{1}{2})v, (x, \varphi)) \\ &\quad + 2xY(L(0)v, (x, \varphi)) + x\varphi Y(G(-\frac{1}{2})v, (x, \varphi)) \\ &\quad + x^2Y(L(-1)v, (x, \varphi)). \end{aligned} \quad (46)$$

Making repeated use of properties (36) - (43), we have

$$e^{x_0L(-1)}Y(v, (x, \varphi))e^{-x_0L(-1)} = Y(e^{x_0L(-1)}v, (x, \varphi)) \quad (47)$$

$$= Y(v, (x + x_0, \varphi)) \quad (48)$$

$$e^{\varphi_0G(-\frac{1}{2})}Y(v, (x, \varphi))e^{-\varphi_0G(-\frac{1}{2})} = Y(e^{\varphi_0G(-\frac{1}{2})-2\varphi_0\varphi L(-1)}v, (x, \varphi)) \quad (49)$$

$$= Y(v, (x + \varphi\varphi_0, \varphi + \varphi_0)), \quad (50)$$

and thus

$$\begin{aligned} e^{x_0L(-1)+\varphi_0G(-\frac{1}{2})}Y(v, (x, \varphi))e^{-x_0L(-1)-\varphi_0G(-\frac{1}{2})} &= \\ &= Y(e^{x_0L(-1)+\varphi_0G(-\frac{1}{2})-2\varphi_0\varphi L(-1)}v, (x, \varphi)) \end{aligned} \quad (51)$$

$$= Y(v, (x + x_0 + \varphi\varphi_0, \varphi + \varphi_0)). \quad (52)$$

From the creation property and (40), we have

$$e^{xL(-1)+\varphi G(-\frac{1}{2})}v = Y(v, (x, \varphi))\mathbf{1}. \quad (53)$$

Note that the right-hand side of the Jacobi identity is invariant under

$$(u, v, x_0, x_1, x_2, \varphi_1, \varphi_2) \longleftrightarrow ((-1)^{\eta(v)\eta(u)}v, u, -x_0, x_2, x_1, \varphi_2, \varphi_1).$$

Thus the left-hand side of the Jacobi identity must be symmetric with respect to this also. Then using (53), we have the following *skew-supersymmetry* property: for  $u, v$  of homogeneous sign in  $V$

$$e^{xL(-1)+\varphi G(-\frac{1}{2})}Y(v, (-x, -\varphi))u = (-1)^{\eta(v)\eta(u)}Y(u, (x, \varphi))v. \quad (54)$$

Let  $(V_1, Y_1(\cdot, (x, \varphi)), \mathbf{1}_1, \tau_1)$  and  $(V_2, Y_2(\cdot, (x, \varphi)), \mathbf{1}_2, \tau_2)$  be two vertex operator superalgebras over  $\Lambda_*$ . A *homomorphism of vertex operator superalgebras with odd formal variables* is a doubly graded  $\Lambda_*$ -module homomorphism  $\gamma : V_1 \longrightarrow V_2$  (i.e.,  $\gamma : (V_1)_{(n)}^i \longrightarrow (V_2)_{(n)}^i$  for  $n \in \frac{1}{2}\mathbb{Z}$ , and  $i \in \mathbb{Z}_2$ ) such that

$$\gamma(Y_1(u, (x, \varphi))v) = Y_2(\gamma(u), (x, \varphi))\gamma(v) \quad \text{for } u, v \in V_1,$$

$$\gamma(\mathbf{1}_1) = \mathbf{1}_2, \text{ and } \gamma(\tau_1) = \tau_2.$$

**Remark 4.2** In  $N = 1$  superconformal field theory there is an inherent choice of square root being made in the choice of superderivation  $D$  satisfying  $D^2 = \frac{\partial}{\partial z}$  and correspondingly in the choice of Neveu-Schwarz algebra representative element  $G(-\frac{1}{2})$  satisfying  $G(-\frac{1}{2})^2 = L(-1)$ . In each case, if  $D$  satisfies  $D^2 = \frac{\partial}{\partial z}$ , then so does  $-D$ , and if  $G(-\frac{1}{2})$  satisfies  $G(-\frac{1}{2})^2 = L(-1)$ , then so does  $-G(-\frac{1}{2})$ . The transformation  $D \leftrightarrow -D$  actually corresponds to the transformation  $\varphi \leftrightarrow -\varphi$ , and algebraically to the transformation  $G(-\frac{1}{2}) \leftrightarrow -G(-\frac{1}{2})$ . This symmetry gives an isomorphism of vertex operator superalgebras with odd formal variables given by  $(V, Y(\cdot, (x, \varphi)), \mathbf{1}, \tau) \leftrightarrow (V, Y(\cdot, (x, -\varphi)), \mathbf{1}, -\tau)$ .

## 5 Vertex operator superalgebras over $\Lambda_*$ and without odd formal variables

**Definition 5.1** A  $(N = 1 \text{ Neveu-Schwarz})$  vertex operator superalgebra over  $\Lambda_*$  and without odd variables is a  $\frac{1}{2}\mathbb{Z}$ -graded (by weight)  $\Lambda_*$ -module which is also  $\mathbb{Z}_2$ -graded (by sign)

$$V = \coprod_{n \in \frac{1}{2}\mathbb{Z}} V_{(n)} = \coprod_{n \in \frac{1}{2}\mathbb{Z}} V_{(n)}^0 \oplus \coprod_{n \in \frac{1}{2}\mathbb{Z}} V_{(n)}^1 = V^0 \oplus V^1$$

such that

$$\dim V_{(n)} < \infty \quad \text{for } n \in \frac{1}{2}\mathbb{Z},$$

$$V_{(n)} = 0 \quad \text{for } n \text{ sufficiently small,}$$

equipped with a linear map  $V \otimes V \longrightarrow V[[x, x^{-1}]]$ , or equivalently,

$$\begin{aligned} V &\longrightarrow (\text{End } V)[[x, x^{-1}]] \\ v &\mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \end{aligned}$$

where  $v_n \in (\text{End } V)^{\eta(v)}$  for  $v$  of homogeneous sign in  $V$ ,  $x$  is an even formal variable, and  $Y(v, x)$  denotes the *vertex operator associated with  $v$* , and equipped also with two distinguished homogeneous vectors  $\mathbf{1} \in V_{(0)}^0$  (the *vacuum*) and  $\tau \in V_{(\frac{3}{2})}^1$  (the *Neveu-Schwarz element*). The following conditions are assumed for  $u, v \in V$ :

$$u_n v = 0 \quad \text{for } n \in \mathbb{Z} \text{ sufficiently large;}$$

$$Y(\mathbf{1}, x) = 1 \quad (1 \text{ on the right being the identity operator});$$

the *creation property* holds:

$$Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x \rightarrow 0} Y(v, x)\mathbf{1} = v;$$

the *Jacobi identity* holds:

$$\begin{aligned} x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1) Y(v, x_2) \\ - (-1)^{\eta(u)\eta(v)} x_0^{-1} \delta\left(\frac{-x_2 + x_1}{x_0}\right) Y(v, x_2) Y(u, x_1) \\ = x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u, x_0)v, x_2), \end{aligned}$$

for  $u, v$  of homogeneous sign in  $V$ ; the Neveu-Schwarz algebra relations hold:

$$\begin{aligned} [L(m), L(n)] &= (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}(\text{rank } V), \\ \left[G\left(m + \frac{1}{2}\right), L(n)\right] &= \left(m - \frac{n-1}{2}\right)G\left(m + n + \frac{1}{2}\right), \\ \left[G\left(m + \frac{1}{2}\right), G\left(n - \frac{1}{2}\right)\right] &= 2L(m + n) + \frac{1}{3}(m^2 + m)\delta_{m+n,0}(\text{rank } V), \end{aligned}$$



for  $m, n \in \mathbb{Z}$ , where

$$G(n + \frac{1}{2}) = \tau_{n+1} \quad \text{for } n \in \mathbb{Z}, \text{ i.e., } Y(\tau, x) = \sum_{n \in \mathbb{Z}} G(n + \frac{1}{2}) x^{-n - \frac{1}{2} - \frac{3}{2}},$$

and  $\text{rank } V \in \mathbb{C}$ ;

$$L(0)v = nv \quad \text{for } n \in \frac{1}{2}\mathbb{Z} \quad \text{and } v \in V_{(n)};$$

$$\frac{\partial}{\partial x} Y(v, x) = Y(L(-1)v, x). \quad (55)$$

The vertex operator superalgebra just defined is denoted by

$$(V, Y(\cdot, x), \mathbf{1}, \tau).$$

Some consequences of the definition are

$$[L(-1), Y(v, x)] = Y(L(-1)v, x); \quad (56)$$

$$[G(-\frac{1}{2}), Y(v, x)] = Y(G(-\frac{1}{2})v, x); \quad (57)$$

$$L(n)\mathbf{1} = 0, \text{ and } G(n + \frac{1}{2})\mathbf{1} = 0, \text{ for } n \geq -1. \quad (58)$$

Note that our definition of vertex operator superalgebra over  $\Lambda_*$  (without formal variables) is an obvious extension of the usual notion of vertex operator algebra [Bo], [FLM], [FHL] and vertex operator superalgebra (cf. [T], [G], [FFR], [DL], and [KW]). For instance, any of the examples of “ $N = 1$  NS-type SVOAs” can be extended in the obvious way to be a  $\Lambda_*$ -module instead of just a vector space over  $\mathbb{C}$ , thus giving an  $N = 1$  Neveu-Schwarz vertex operator over  $\Lambda_*$  as defined above. As we shall see in the next section, the categories of  $N = 1$  Neveu-Schwarz vertex operator superalgebras over  $\Lambda_*$  with and without odd formal variables, respectively, are isomorphic. This isomorphism can be used to give examples of  $N = 1$  Neveu-Schwarz vertex operator algebras over  $\Lambda_*$  with odd formal variables from the examples of  $N = 1$  NS-type SVOAs.

Let  $(V_1, Y_1(\cdot, x), \mathbf{1}_1, \tau_1)$  and  $(V_2, Y_2(\cdot, x), \mathbf{1}_2, \tau_2)$  be two vertex operator superalgebras over  $\Lambda_*$ . A *homomorphism of vertex operator superalgebras without odd formal variables* is a doubly graded  $\Lambda_*$ -module homomorphism  $\gamma : V_1 \longrightarrow V_2$  such that

$$\gamma(Y_1(u, x)v) = Y_2(\gamma(u), x)\gamma(v) \quad \text{for } u, v \in V_1,$$

$$\gamma(\mathbf{1}_1) = \mathbf{1}_2, \text{ and } \gamma(\tau_1) = \tau_2.$$

## 6 The isomorphism between the category of vertex operator superalgebras with odd formal variables and the category of vertex operator superalgebras without odd formal variables

**Proposition 6.1** *Let  $(V, Y(\cdot, (x, \varphi)), \mathbf{1}, \tau)$  be a vertex operator superalgebra with odd formal variables. Then  $(V, Y(\cdot, (x, 0)), \mathbf{1}, \tau)$  is a vertex operator superalgebra without odd formal variables.*

*Proof:* It is trivial that  $(V, Y(\cdot, (x, 0)), \mathbf{1}, \tau)$  satisfies all the axioms for a vertex operator superalgebra without odd formal variables except for the  $L(-1)$ -derivative property (55). But this follows as a consequence of the definition of a vertex operator superalgebra with odd formal variables (29).  $\square$

Let  $(V, Y(\cdot, x), \mathbf{1}, \tau)$  be a vertex operator superalgebra without odd formal variables. Define

$$\tilde{Y}(v, (x, \varphi)) = Y(v, x) + \varphi Y(G(-\frac{1}{2})v, x).$$

**Proposition 6.2**  *$(V, \tilde{Y}(\cdot, (x, \varphi)), \mathbf{1}, \tau)$  is a vertex operator superalgebra with odd formal variables.*

*Proof:* Axioms (16), (17), (18), and (21) are obvious. Axiom (19) holds since by consequence (58) of the definition of  $(V, Y(\cdot, x), \mathbf{1}, \tau)$ , we have

$$\tilde{Y}(\mathbf{1}, (x, \varphi)) = Y(\mathbf{1}, x) + \varphi Y(G(-\frac{1}{2})\mathbf{1}, x) = Y(\mathbf{1}, x) + \varphi Y(0, x) = \mathbf{1}.$$

The creation property holds since by the creation property for  $(V, Y(\cdot, x), \mathbf{1}, \tau)$

$$\tilde{Y}(v, (x, \varphi))\mathbf{1} = Y(v, x)\mathbf{1} + \varphi Y(G(-\frac{1}{2})v, x)\mathbf{1} \in V[[x]][\varphi],$$

and

$$\begin{aligned} \lim_{(x, \varphi) \rightarrow 0} \tilde{Y}(v, (x, \varphi))\mathbf{1} &= \lim_{(x, \varphi) \rightarrow 0} Y(v, x)\mathbf{1} + \lim_{(x, \varphi) \rightarrow 0} \varphi Y(G(-\frac{1}{2})v, x)\mathbf{1} \\ &= \lim_{x \rightarrow 0} Y(v, x)\mathbf{1} = v. \end{aligned}$$

To prove the Jacobi identity for  $(V, \tilde{Y}(\cdot, (x, \varphi)), \mathbf{1}, \tau)$ , we use the Jacobi identity for  $(V, Y(\cdot, x), \mathbf{1}, \tau)$  and the  $L(-1)$ - and  $G(-\frac{1}{2})$ -bracket properties for  $(V, Y(\cdot, x), \mathbf{1}, \tau)$  given by (56) and (57).

$$\begin{aligned}
& x_0^{-1} \delta \left( \frac{x_1 - x_2 - \varphi_1 \varphi_2}{x_0} \right) \tilde{Y}(u, (x_1, \varphi_1)) \tilde{Y}(v, (x_2, \varphi_2)) \\
& \quad - (-1)^{\eta(u)\eta(v)} x_0^{-1} \delta \left( \frac{x_2 - x_1 + \varphi_1 \varphi_2}{-x_0} \right) \tilde{Y}(v, (x_2, \varphi_2)) \tilde{Y}(u, (x_1, \varphi_1)) \\
= & \quad x_0^{-1} \delta \left( \frac{x_1 - x_2 - \varphi_1 \varphi_2}{x_0} \right) \left( Y(u, x_1) + \varphi_1 Y(G(-\frac{1}{2})u, x_1) \right) \\
& \quad \left( Y(v, x_2) + \varphi_2 Y(G(-\frac{1}{2})v, x_2) \right) \\
& \quad - (-1)^{\eta(u)\eta(v)} x_0^{-1} \delta \left( \frac{x_2 - x_1 + \varphi_1 \varphi_2}{-x_0} \right) \left( Y(v, x_2) + \varphi_2 Y(G(-\frac{1}{2})v, x_2) \right) \\
& \quad \left( Y(u, x_1) + \varphi_1 Y(G(-\frac{1}{2})u, x_1) \right) \\
= & \quad x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(u, x_1) Y(v, x_2) \\
& \quad - (-1)^{\eta(u)\eta(v)} x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y(v, x_2) Y(u, x_1) \\
& \quad + x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) \varphi_1 Y(G(-\frac{1}{2})u, x_1) Y(v, x_2) \\
& \quad - (-1)^{\eta(u)\eta(v)} x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y(v, x_2) \varphi_1 Y(G(-\frac{1}{2})u, x_1) \\
& \quad + x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(u, x_1) \varphi_2 Y(G(-\frac{1}{2})v, x_2) \\
& \quad - (-1)^{\eta(u)\eta(v)} x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) \varphi_2 Y(G(-\frac{1}{2})v, x_2) Y(u, x_1) \\
& \quad + x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) \varphi_1 Y(G(-\frac{1}{2})u, x_1) \varphi_2 Y(G(-\frac{1}{2})v, x_2) \\
& \quad - (-1)^{\eta(u)\eta(v)} x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) \varphi_2 Y(G(-\frac{1}{2})v, x_2) \varphi_1 Y(G(-\frac{1}{2})u, x_1) \\
& \quad - \varphi_1 \varphi_2 x_0^{-2} \delta' \left( \frac{x_1 - x_2}{x_0} \right) Y(u, x_1) Y(v, x_2) \\
& \quad + \varphi_1 \varphi_2 (-1)^{\eta(u)\eta(v)} x_0^{-2} \delta' \left( \frac{x_2 - x_1}{-x_0} \right) Y(v, x_2) Y(u, x_1) \\
= & \quad x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2)
\end{aligned}$$

$$\begin{aligned}
& + \varphi_1 x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) Y(G(-\frac{1}{2})u, x_1) Y(v, x_2) \\
& - \varphi_1 (-1)^{\eta(u)\eta(v)+\eta(v)} x_0^{-1} \delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v, x_2) Y(G(-\frac{1}{2})u, x_1) \\
& + \varphi_2 (-1)^{\eta(u)} x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1) Y(G(-\frac{1}{2})v, x_2) \\
& - \varphi_2 (-1)^{\eta(u)\eta(v)} x_0^{-1} \delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(G(-\frac{1}{2})v, x_2) Y(u, x_1) \\
& + \varphi_1 \varphi_2 (-1)^{\eta(u)+1} x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) Y(G(-\frac{1}{2})u, x_1) Y(G(-\frac{1}{2})v, x_2) \\
& - \varphi_1 \varphi_2 (-1)^{\eta(u)\eta(v)+\eta(v)} x_0^{-1} \delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(G(-\frac{1}{2})v, x_2) Y(G(-\frac{1}{2})u, x_1) \\
& - \varphi_1 \varphi_2 \frac{\partial}{\partial x_1} \left( x_0^{-2} \delta\left(\frac{x_1 - x_2}{x_0}\right) \right) Y(u, x_1) Y(v, x_2) \\
& + \varphi_1 \varphi_2 (-1)^{\eta(u)\eta(v)} \frac{\partial}{\partial x_1} \left( x_0^{-2} \delta\left(\frac{x_2 - x_1}{-x_0}\right) \right) Y(v, x_2) Y(u, x_1) \\
= & x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u, x_0)v, x_2) \\
& + \varphi_1 x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(G(-\frac{1}{2})u, x_0)v, x_2) \\
& + \varphi_2 (-1)^{\eta(u)} x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u, x_0)G(-\frac{1}{2})v, x_2) \\
& + \varphi_1 \varphi_2 (-1)^{\eta(u)+1} x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(G(-\frac{1}{2})u, x_0)G(-\frac{1}{2})v, x_2) \\
& - \varphi_1 \varphi_2 \frac{\partial}{\partial x_1} \left( x_0^{-2} \delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1) Y(v, x_2) \right. \\
& \quad \left. - (-1)^{\eta(u)\eta(v)} x_0^{-2} \delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v, x_2) Y(u, x_1) \right) \\
& + \varphi_1 \varphi_2 \left( x_0^{-2} \delta\left(\frac{x_1 - x_2}{x_0}\right) \frac{\partial}{\partial x_1} Y(u, x_1) Y(v, x_2) \right. \\
& \quad \left. - (-1)^{\eta(u)\eta(v)} x_0^{-2} \delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v, x_2) \frac{\partial}{\partial x_1} Y(u, x_1) \right) \\
= & x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) \left( Y(Y(u, x_0)v, x_2) + \varphi_1 Y(Y(G(-\frac{1}{2})u, x_0)v, x_2) \right. \\
& \quad \left. + \varphi_2 (-1)^{\eta(u)} Y(Y(u, x_0)G(-\frac{1}{2})v, x_2) \right)
\end{aligned}$$

$$\begin{aligned}
& + \varphi_1 \varphi_2 (-1)^{\eta(u)+1} Y(Y(G(-\frac{1}{2})u, x_0)G(-\frac{1}{2})v, x_2) \Big) \\
& - \varphi_1 \varphi_2 \frac{\partial}{\partial x_1} \left( x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2) \right) \\
& + \varphi_1 \varphi_2 \left( x_0^{-2} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(L(-1)u, x_1) Y(v, x_2) \right. \\
& \quad \left. - (-1)^{\eta(u)\eta(v)} x_0^{-2} \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y(v, x_2) Y(L(-1)u, x_1) \right) \\
= & x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) \left( Y(Y(u, x_0)v, x_2) + \varphi_1 Y(Y(G(-\frac{1}{2})u, x_0)v, x_2) \right. \\
& + \varphi_2 \left( (-1)^{\eta(u)} Y(Y(u, x_0)G(-\frac{1}{2})v, x_2) + Y \left( \left[ G(-\frac{1}{2}), Y(u, x_0) \right] v, x_2 \right) \right. \\
& \quad \left. \left. - Y(Y(G(-\frac{1}{2})u, x_0)v, x_2) \right) \right. \\
& + \varphi_1 \varphi_2 \left( (-1)^{\eta(u)+1} Y(Y(G(-\frac{1}{2})u, x_0)G(-\frac{1}{2})v, x_2) \right. \\
& \quad \left. \left. + Y(Y(L(-1)u, x_0)v, x_2) \right) \right) \\
& - \varphi_1 \varphi_2 x_2^{-2} \delta' \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2) \\
= & x_2^{-1} \delta \left( \frac{x_1 - x_0 - \varphi_1 \varphi_2}{x_2} \right) \left( Y(Y(u, x_0)v, x_2) + \varphi_1 Y(Y(G(-\frac{1}{2})u, x_0)v, x_2) \right. \\
& + \varphi_2 Y(G(-\frac{1}{2})Y(u, x_0)v, x_2) - \varphi_2 Y(Y(G(-\frac{1}{2})u, x_0)v, x_2) \\
& + \varphi_1 \varphi_2 \left( (-1)^{\eta(u)+1} Y(Y(G(-\frac{1}{2})u, x_0)G(-\frac{1}{2})v, x_2) \right. \\
& \quad \left. \left. + Y \left( \left[ G(-\frac{1}{2}), Y(G(-\frac{1}{2})u, x_0) \right] v, x_2 \right) \right) \right) \\
= & x_2^{-1} \delta \left( \frac{x_1 - x_0 - \varphi_1 \varphi_2}{x_2} \right) \left( Y \left( Y(u, x_0)v + \varphi_1 Y(G(-\frac{1}{2})u, x_0)v \right. \right. \\
& \left. \left. - \varphi_2 Y(G(-\frac{1}{2})u, x_0)v, x_2 \right) + \varphi_2 Y(G(-\frac{1}{2})Y(u, x_0)v, x_2) \right. \\
& \quad \left. + \varphi_1 \varphi_2 Y(G(-\frac{1}{2})Y(G(-\frac{1}{2})u, x_0)v, x_2) \right)
\end{aligned}$$

$$\begin{aligned}
&= x_2^{-1} \delta\left(\frac{x_1 - x_0 - \varphi_1 \varphi_2}{x_2}\right) \left( Y\left(\tilde{Y}(u, (x_0, \varphi_1 - \varphi_2))v, x_2\right) \right. \\
&\quad \left. + \varphi_2 Y\left(G\left(-\frac{1}{2}\right)\tilde{Y}(u, (x_0, \varphi_1 - \varphi_2))v, x_2\right) \right) \\
&= x_2^{-1} \delta\left(\frac{x_1 - x_0 - \varphi_1 \varphi_2}{x_2}\right) \tilde{Y}(\tilde{Y}(u, (x_0, \varphi_1 - \varphi_2))v, (x_2, \varphi_2))
\end{aligned}$$

which gives the Jacobi identity for  $(V, \tilde{Y}(\cdot, (x, \varphi)), \mathbf{1}, \tau)$ .

For the Neveu-Schwarz element  $\tau$ , we have

$$\begin{aligned}
\tilde{Y}(\tau, (x, \varphi)) &= Y(\tau, x) + \varphi Y\left(G\left(-\frac{1}{2}\right)\tau, x\right) \\
&= Y(\tau, x) + \varphi \left[ G\left(-\frac{1}{2}\right), Y(\tau, x) \right] \\
&= \sum_{n \in \mathbb{Z}} G\left(n + \frac{1}{2}\right) x^{-n-2} + \varphi \sum_{n \in \mathbb{Z}} \left[ G\left(-\frac{1}{2}\right), G\left(n + \frac{1}{2}\right) \right] x^{-n-2} \\
&= \sum_{n \in \mathbb{Z}} G\left(n + \frac{1}{2}\right) x^{-n-2} + \varphi \sum_{n \in \mathbb{Z}} 2L(n) x^{-n-2}
\end{aligned}$$

which gives (20). Finally, using the  $L(-1)$ -derivative property for  $(V, Y(\cdot, x), \mathbf{1}, \tau)$ , we have

$$\begin{aligned}
\left(\frac{\partial}{\partial \varphi} + \varphi \frac{\partial}{\partial x}\right) \tilde{Y}(v, (x, \varphi)) &= \left(\frac{\partial}{\partial \varphi} + \varphi \frac{\partial}{\partial x}\right) \left( Y(v, x) + \varphi Y\left(G\left(-\frac{1}{2}\right)v, x\right) \right) \\
&= \varphi \frac{\partial}{\partial x} Y(v, x) + Y\left(G\left(-\frac{1}{2}\right)v, x\right) \\
&= Y\left(G\left(-\frac{1}{2}\right)v, x\right) + \varphi Y(L(-1)v, x) \\
&= Y\left(G\left(-\frac{1}{2}\right)v, x\right) + \varphi Y\left(G\left(-\frac{1}{2}\right)^2 v, x\right) \\
&= \tilde{Y}\left(G\left(-\frac{1}{2}\right)v, (x, \varphi)\right)
\end{aligned}$$

which gives the  $G(-\frac{1}{2})$ -derivative property (22). Thus  $(V, \tilde{Y}(\cdot, (x, \varphi)), \mathbf{1}, \tau)$  is a vertex operator superalgebra with odd formal variables.  $\square$

Let  $\mathbf{SV}(\varphi, c, *)$  denote the category of vertex operator superalgebras over  $\Lambda_*$  with odd formal variables and with central charge, i.e., rank,  $c \in \mathbb{C}$ ,

and let  $\mathbf{SV}(c, *)$  denote the category of vertex operator superalgebras over  $\bigwedge_*$  without odd formal variables and with central charge, i.e., rank,  $c \in \mathbb{C}$ . Let  $1_{SV_\varphi}$  and  $1_{SV}$  be the identity functors on the categories  $\mathbf{SV}(\varphi, c, *)$  and  $\mathbf{SV}(c, *)$ , respectively.

**Theorem 6.3** *For any  $c \in \mathbb{C}$ , the two categories  $\mathbf{SV}(\varphi, c, *)$  and  $\mathbf{SV}(c, *)$  are isomorphic. That is there exist two functors  $F_0 : \mathbf{SV}(\varphi, c, *) \longrightarrow \mathbf{SV}(c, *)$  and  $F_\varphi : \mathbf{SV}(c, *) \longrightarrow \mathbf{SV}(\varphi, c, *)$  such that  $F_0 \circ F_\varphi = 1_{SV}$  and  $F_\varphi \circ F_0 = 1_{SV_\varphi}$ .*

*Proof:* We first define  $F_0$  by

$$F_0(V, Y(\cdot, (x, \varphi)), \mathbf{1}, \tau) = (V, Y(\cdot, (x, 0)), \mathbf{1}, \tau), \quad \text{and} \quad F_0(\gamma) = \gamma.$$

Proposition 6.1 shows that  $F_0$  takes objects in  $\mathbf{SV}(\varphi, c, *)$  to objects in  $\mathbf{SV}(c, *)$ . It is clear that  $F_0$  takes morphisms in  $\mathbf{SV}(\varphi, c, *)$  to morphisms in  $\mathbf{SV}(c, *)$  and that  $F_0$  is a functor.

We next define  $F_\varphi$  by

$$F_\varphi(V, Y(\cdot, x), \mathbf{1}, \tau) = (V, \tilde{Y}(\cdot, (x, \varphi)), \mathbf{1}, \tau), \quad \text{and} \quad F_\varphi(\gamma) = \gamma$$

where  $\tilde{Y}(v, (x, \varphi)) = Y(v, x) + \varphi Y(G(-\frac{1}{2})v, x)$ . Proposition 6.2 shows that  $F_\varphi$  takes objects in  $\mathbf{SV}(c, *)$  to objects in  $\mathbf{SV}(\varphi, c, *)$ . Let

$$\gamma : (V_1, Y_1(\cdot, x), \mathbf{1}_1, \tau_1) \longrightarrow (V_2, Y_2(\cdot, x), \mathbf{1}_2, \tau_2)$$

be a homomorphism of vertex operator superalgebras without odd formal variables. Then  $\gamma(\tau_1) = \tau_2$ . Denoting  $(\tau_1)_0 = G_1(-\frac{1}{2})$  and  $(\tau_2)_0 = G_2(-\frac{1}{2})$ , we have  $\gamma(G_1(-\frac{1}{2})u) = G_2(-\frac{1}{2})\gamma(u)$ . Thus

$$\begin{aligned} \gamma(\tilde{Y}_1(u, (x, \varphi))v) &= \gamma(Y_1(u, x)v + \varphi Y_1(G_1(-\frac{1}{2})u, x)v) \\ &= \gamma(Y_1(u, x)v) + \varphi \gamma(Y_1(G_1(-\frac{1}{2})u, x)v) \\ &= Y_2(\gamma(u), x)\gamma(v) + \varphi Y_1(\gamma(G_1(-\frac{1}{2})u), x)\gamma(v) \\ &= Y_2(\gamma(u), x)\gamma(v) + \varphi Y_1(G_2(-\frac{1}{2})\gamma(u), x)\gamma(v) \\ &= \tilde{Y}_2(\gamma(u), (x, \varphi))\gamma(v) \end{aligned}$$

which shows that  $F_\varphi(\gamma) = \gamma$  is a homomorphism of vertex operator superalgebras with odd formal variables, i.e.,  $F_\varphi$  takes morphisms in  $\mathbf{SV}(c, *)$  to morphisms in  $\mathbf{SV}(\varphi, c, *)$ . It is clear that  $F_\varphi$  is a functor.

The fact that  $F_0 \circ F_\varphi = 1_{SV}$  and  $F_\varphi \circ F_0 = 1_{SV_\varphi}$  on morphisms is trivial. The fact that  $F_0 \circ F_\varphi = 1_{SV}$  on objects is given by

$$Y(v, (x, \varphi)) = Y(v, (x, 0)) + \varphi Y(G(-\frac{1}{2})v, (x, 0)),$$

and the fact that  $F_\varphi \circ F_0 = 1_{SV_\varphi}$  on objects is given by

$$Y(v, x) = Y(v, x) + \varphi Y(G(-\frac{1}{2})v, x) \Big|_{\varphi=0}.$$

□

## 7 Weak supercommutativity and weak associativity for vertex operator superalgebras with odd formal variables

In this section we show that the properties which we call “weak” supercommutativity and “weak” associativity for vertex operator superalgebras without odd formal variables, as formulated and studied for instance in [DL], [L1] and [L2], have the expected analogues when we add the odd formal variables. We refer to these properties as “weak” supercommutativity and “weak” associativity because in Section 9, we will prove slightly stronger statements about the nature of certain rational functions associated with products and iterates of vertex operators. We also note that we could prove Propositions 7.1, 7.2 and 7.3 by using weak commutativity, weak associativity and their equivalence with the Jacobi identity, respectively, for a vertex operator superalgebra without odd formal variables and then by using Theorem 6.3. However, we choose to prove these properties directly using the definition of vertex operator superalgebra with odd formal variables.

In this section we are following and extending the corresponding results and arguments of [L1] and [L2].

Let  $\mathbb{Z}_+$  denote the positive integers.

**Proposition 7.1 (weak supercommutativity)** *Let  $(V, Y(\cdot, (x, \varphi)), \mathbf{1}, \tau)$  be a vertex operator algebra with odd formal variables and  $u, v \in V$  with homogeneous sign. Then there exists  $k \in \mathbb{Z}_+$  such that*

$$(x_1 - x_2 - \varphi_1 \varphi_2)^k Y(u, (x_1, \varphi_1)) Y(v, (x_2, \varphi_2)) =$$



$$(-1)^{\eta(u)\eta(v)}(x_1 - x_2 - \varphi_1\varphi_2)^k Y(v, (x_2, \varphi_2))Y(u, (x_1, \varphi_1)).$$

Furthermore this weak supercommutativity follows from the truncation condition (18) and the Jacobi identity.

*Proof:* Let  $m \in \mathbb{Z}_+$ . Taking  $\text{Res}_{x_0} x_0^m$  of the Jacobi identity, we have

$$\begin{aligned} & (x_1 - x_2 - \varphi_1\varphi_2)^m [Y(u, (x_1, \varphi_1)), Y(v, (x_2, \varphi_2))] = \\ &= (x_1 - x_2 - \varphi_1\varphi_2)^m \left( Y(u, (x_1, \varphi_1))Y(v, (x_2, \varphi_2)) \right. \\ & \quad \left. - (-1)^{\eta(u)\eta(v)} Y(v, (x_2, \varphi_2))Y(u, (x_1, \varphi_1)) \right) \\ &= \text{Res}_{x_0} x_2^{-1} \delta \left( \frac{x_1 - x_0 - \varphi_1\varphi_2}{x_2} \right) x_0^m Y(Y(u, x_0, \varphi_1 - \varphi_2)v, (x_2, \varphi_2)) \\ &= \text{Res}_{x_0} x_1^{-1} \delta \left( \frac{x_2 + x_0 + \varphi_1\varphi_2}{x_1} \right) x_0^m Y(Y(u, x_0, \varphi_1 - \varphi_2)v, (x_2, \varphi_2)) \\ &= \sum_{n \in \mathbb{N}} \frac{1}{n!} \left( \left( \frac{\partial}{\partial x_2} \right)^n x_1^{-1} \delta \left( \frac{x_2}{x_1} \right) \right) Y((u_{n+m} + (\varphi_1 - \varphi_2)u_{n+m-\frac{1}{2}})v, (x_2, \varphi_2)) \\ & \quad + \varphi_1\varphi_2 \sum_{n \in \mathbb{N}} \frac{1}{n!} \left( \left( \frac{\partial}{\partial x_2} \right)^{n+1} x_1^{-1} \delta \left( \frac{x_2}{x_1} \right) \right) \\ & \quad Y((u_{n+m-1} + (\varphi_1 - \varphi_2)u_{n+m-\frac{3}{2}})v, (x_2, \varphi_2)). \end{aligned}$$

Let  $k \in \mathbb{Z}_+$  be such that  $u_l v = 0$  for all  $l \in \frac{1}{2}\mathbb{Z}_+$ ,  $l \geq k - \frac{3}{2}$ . Setting  $m = k$ , we obtain weak supercommutativity.  $\square$

**Proposition 7.2 (weak associativity)** *Let  $(V, Y(\cdot, (x, \varphi)), \mathbf{1}, \tau)$  be a vertex operator algebra with odd formal variables and  $u, v \in V$  with homogeneous sign. Then there exists  $k \in \mathbb{Z}_+$  such that for any  $w \in V$*

$$\begin{aligned} & (x_0 + x_2 + \varphi_1\varphi_2)^k Y(Y(u, (x_0, \varphi_1 - \varphi_2)v, (x_2, \varphi_2))w = \\ & (x_0 + x_2 + \varphi_1\varphi_2)^k Y(u, (x_0 + x_2 + \varphi_1\varphi_2, \varphi_1))Y(v, (x_2, \varphi_2))w \end{aligned}$$

Furthermore this weak associativity follows from the truncation condition (18) and the Jacobi identity.

*Proof:* Taking  $\text{Res}_{x_1}$  of the Jacobi identity, we obtain the following iterate

$$Y(Y(u, (x_0, \varphi_1 - \varphi_2)v, (x_2, \varphi_2))) =$$

$$\begin{aligned}
&= \text{Res}_{x_1} x_1^{-1} \delta\left(\frac{x_2 + x_0 + \varphi_1 \varphi_2}{x_1}\right) Y(Y(u, (x_0, \varphi_1 - \varphi_2)v, (x_2, \varphi_2))) \\
&= \text{Res}_{x_1} x_2^{-1} \delta\left(\frac{x_1 - x_0 - \varphi_1 \varphi_2}{x_2}\right) Y(Y(u, (x_0, \varphi_1 - \varphi_2)v, (x_2, \varphi_2))) \\
&= \text{Res}_{x_1} \left( x_0^{-1} \delta\left(\frac{x_1 - x_2 - \varphi_1 \varphi_2}{x_0}\right) Y(u, (x_1, \varphi_1)) Y(v, (x_2, \varphi_2)) \right. \\
&\quad \left. - (-1)^{\eta(u)\eta(v)} x_0^{-1} \delta\left(\frac{x_2 - x_1 + \varphi_1 \varphi_2}{-x_0}\right) Y(v, (x_2, \varphi_2)) Y(u, (x_1, \varphi_1)) \right) \\
&= \text{Res}_{x_1} \left( x_1^{-1} \delta\left(\frac{x_0 + x_2 + \varphi_1 \varphi_2}{x_1}\right) Y(u, (x_1, \varphi_1)) Y(v, (x_2, \varphi_2)) \right. \\
&\quad \left. - (-1)^{\eta(u)\eta(v)} Y(v, (x_2, \varphi_2)) \left( x_0^{-1} \delta\left(\frac{x_1 - x_2 - \varphi_1 \varphi_2}{x_0}\right) \right. \right. \\
&\quad \left. \left. - x_2^{-1} \delta\left(\frac{x_1 - x_0 - \varphi_1 \varphi_2}{x_2}\right) \right) Y(u, (x_1, \varphi_1)) \right) \\
&= Y(u, (x_0 + x_2 + \varphi_1 \varphi_2, \varphi_1)) Y(v, (x_2, \varphi_2)) \\
&\quad - (-1)^{\eta(u)\eta(v)} Y(v, (x_2, \varphi_2)) \left( Y(u, (x_0 + x_2 + \varphi_1 \varphi_2, \varphi_1)) \right. \\
&\quad \left. - Y(u, (x_2 + x_0 + \varphi_1 \varphi_2, \varphi_1)) \right).
\end{aligned}$$

For any  $w \in V$ , let  $k \in \mathbb{Z}_+$  be such that  $x^k Y(u, (x, \varphi))w$  involves only positive powers of  $x$ . Then

$$\begin{aligned}
&(x_0 + x_2 + \varphi_1 \varphi_2)^k \left( Y(u, (x_0 + x_2 + \varphi_1 \varphi_2, \varphi_1)) \right. \\
&\quad \left. - Y(u, (x_2 + x_0 + \varphi_1 \varphi_2, \varphi_1)) \right) w = 0
\end{aligned}$$

and weak associativity follows.  $\square$

**Proposition 7.3** *In the presence of the other axioms in the definition of vertex operator superalgebra with odd formal variables, the Jacobi identity is equivalent to weak supercommutativity and weak associativity.*

*Proof:* Propositions 7.1 and 7.2 show that in the presence of the other axioms for a vertex operator superalgebra with odd formal variables, the Jacobi identity implies weak supercommutativity and weak associativity.

Assume weak supercommutativity and weak associativity hold. Choose  $k \in \mathbb{Z}_+$  such that  $u_mv = u_mw = 0$  for all  $m \in \frac{1}{2}\mathbb{Z}_+$ ,  $m \geq k$ . Then

$$\begin{aligned}
& x_0^k x_1^k \left( x_0^{-1} \delta \left( \frac{x_1 - x_2 - \varphi_1 \varphi_2}{x_0} \right) Y(u, (x_1, \varphi_1)) Y(v, (x_2, \varphi_2)) w \right. \\
& \quad \left. - (-1)^{\eta(u)\eta(v)} x_0^{-1} \delta \left( \frac{x_2 - x_1 + \varphi_1 \varphi_2}{-x_0} \right) Y(v, (x_2, \varphi_2)) Y(u, (x_1, \varphi_1)) w \right) \\
&= x_0^{-1} \delta \left( \frac{x_1 - x_2 - \varphi_1 \varphi_2}{x_0} \right) x_1^k (x_1 - x_2 - \varphi_1 \varphi_2)^k \\
& \quad Y(u, (x_1, \varphi_1)) Y(v, (x_2, \varphi_2)) w \\
& \quad - (-1)^{\eta(u)\eta(v)} x_0^{-1} \delta \left( \frac{x_2 - x_1 + \varphi_1 \varphi_2}{-x_0} \right) x_1^k (x_1 - x_2 - \varphi_1 \varphi_2)^k \\
& \quad Y(v, (x_2, \varphi_2)) Y(u, (x_1, \varphi_1)) w \\
&= x_2^{-1} \delta \left( \frac{x_1 - x_0 - \varphi_1 \varphi_2}{x_2} \right) \left( x_1^k (x_1 - x_2 - \varphi_1 \varphi_2)^k \right. \\
& \quad \left. Y(u, (x_1, \varphi_1)) Y(v, (x_2, \varphi_2)) w \right).
\end{aligned} \tag{59}$$

By weak supercommutativity,  $x_1^k (x_1 - x_2 - \varphi_1 \varphi_2)^k Y(u, (x_1, \varphi_1)) Y(v, (x_2, \varphi_2)) w$  involves only nonnegative powers of  $x_1$ , and  $u_m w = 0$  for  $m \geq k$ . Thus in this case, we can replace  $x_1$  by  $x_2 + x_0 + \varphi_1 \varphi_2$  or  $x_0 + x_2 + \varphi_1 \varphi_2$ . Therefore (59) is equal to

$$\begin{aligned}
& x_2^{-1} \delta \left( \frac{x_1 - x_0 - \varphi_1 \varphi_2}{x_2} \right) \left( x_0^k (x_0 + x_2 + \varphi_1 \varphi_2)^k \right. \\
& \quad \left. Y(u, (x_0 + x_2 + \varphi_1 \varphi_2, \varphi_1)) Y(v, (x_2, \varphi_2)) w \right) \\
&= x_2^{-1} \delta \left( \frac{x_1 - x_0 - \varphi_1 \varphi_2}{x_2} \right) \left( x_0^k (x_0 + x_2 + \varphi_1 \varphi_2)^k \right. \\
& \quad \left. Y(Y(u, (x_0, \varphi_1 - \varphi_2)) v, (x_2, \varphi_2)) w \right) \\
&= x_2^{-1} \delta \left( \frac{x_1 - x_0 - \varphi_1 \varphi_2}{x_2} \right) \left( x_0^k x_1^k Y(Y(u, (x_0, \varphi_1 - \varphi_2)) v, (x_2, \varphi_2)) w \right) \\
&= x_0^k x_1^k x_2^{-1} \delta \left( \frac{x_1 - x_0 - \varphi_1 \varphi_2}{x_2} \right) Y(Y(u, (x_0, \varphi_1 - \varphi_2)) v, (x_2, \varphi_2)) w
\end{aligned}$$

which implies the Jacobi identity.  $\square$

## 8 Expansions of rational superfunctions

In order to formulate the notions of associativity and supercommutativity, we will need to interpret correlation functions of vertex operators with odd formal variables as expansions of certain rational superfunctions. In this section we follow and extend the treatment of rational functions as presented in [FHL].

Let  $T(U) = \coprod_{n \in \mathbb{N}} T^n(U)$  be the tensor algebra over the vector space  $U$ , where  $T^n(U)$  is the  $n$ -fold tensor product of  $U$ , and let  $\mathcal{J}$  be the ideal of  $T(U)$  generated by the elements  $a \otimes b + b \otimes a$  for  $a, b \in U$ . Then  $\bigwedge(U) = T(U)/\mathcal{J}$ . (It is understood that  $T^0(U) = \mathbb{C}$ .) Let  $\pi_B$  be the projection from  $T(U)$  onto  $T^0(U)$ . Then  $\pi_B$  is well defined on  $\bigwedge(U)$  and is called the projection onto the *body* of  $\bigwedge(U)$  (cf. [D], [Ba1]). For  $a \in \bigwedge_*$ , denote  $\pi_B(a) = a_B$ .

Let  $\bigwedge_*[x_1, x_2, \dots, x_n]_S$  be the ring of rational functions obtained by inverting (localizing with respect to) the set

$$S = \left\{ \sum_{i=1}^n a_i x_i : a_i \in \bigwedge_*^0, \text{ not all } (a_i)_B = 0 \right\}.$$

Recall the map  $\iota_{i_1 \dots i_2} : \mathbb{F}[x_1, \dots, x_n]_S \longrightarrow \mathbb{F}[[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]]$  defined in [FLM] where coefficients of elements in  $S$  are restricted to the field  $\mathbb{F}$ . We extend this map to  $\bigwedge_*[x_1, x_2, \dots, x_n]_S[\varphi_1, \varphi_2, \dots, \varphi_n] = \bigwedge_*[x_1, \varphi_1, x_2, \varphi_2, \dots, x_n, \varphi_n]_S$  in the obvious way obtaining

$$\iota_{i_1 \dots i_2} : \bigwedge_*[x_1, \varphi_1, \dots, x_n, \varphi_n]_S \longrightarrow \bigwedge_*[[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]][\varphi_1, \dots, \varphi_n].$$

Let  $\bigwedge_*[x_1, \varphi_1, x_2, \varphi_2, \dots, x_n, \varphi_n]_{S'}$  be the ring of rational functions obtained by inverting the set

$$S' = \left\{ \sum_{\substack{i,j=1 \\ i < j}}^n (a_i x_i + a_{ij} \varphi_i \varphi_j) : a_i, a_{ij} \in \bigwedge_*^0, \text{ not all } (a_i)_B = 0 \right\}.$$

Since we use the convention that a function of even and odd variables should be expanded about the even variables, we have

$$\frac{1}{\sum_{\substack{i,j=1 \\ i < j}}^n (a_i x_i + a_{ij} \varphi_i \varphi_j)} = \frac{1}{\sum_{i=1}^n a_i x_i} - \frac{\sum_{\substack{i,j=1 \\ i < j}}^n a_{ij} \varphi_i \varphi_j}{(\sum_{i=1}^n a_i x_i)^2}.$$

Thus

$$\bigwedge_*[x_1, \varphi_1, x_2, \varphi_2, \dots, x_n, \varphi_n]_{S'} \subseteq \bigwedge_*[x_1, \varphi_1, x_2, \varphi_2, \dots, x_n, \varphi_n]_S,$$

and  $\iota_{i_1 \dots i_2}$  is well defined on  $\bigwedge_*[x_1, \varphi_1, x_2, \varphi_2, \dots, x_n, \varphi_n]_{S'}$ .

In the case  $n = 2$ ,

$$\iota_{12} : \bigwedge_*[x_1, \varphi_1, x_2, \varphi_2]_{S'} \longrightarrow \bigwedge_*[[x_1, x_2]][\varphi_1, \varphi_2]$$

is given by first expanding an element of  $\bigwedge_*[x_1, \varphi_1, x_2, \varphi_2]_{S'}$  as a formal series in  $\bigwedge_*[x_1, \varphi_1, x_2, \varphi_2]_S$  and then expanding each term as a series in  $\bigwedge_*[[x_1, x_2]][\varphi_1, \varphi_2]$  containing at most finitely many negative powers of  $x_2$  (using binomial expansions for negative powers of linear polynomials involving both  $x_1$  and  $x_2$ ).

## 9 Duality for vertex operator superalgebras

In [Ba1], we formulate the notion of  $N = 1$  supergeometric vertex operator superalgebra and show that any such object defines a  $N = 1$  Neveu-Schwarz vertex operator superalgebra with odd formal variables. To show that the alleged vertex operator superalgebra satisfies the Jacobi identity, we need the notions of associativity and supercommutativity for a vertex operator superalgebra with odd formal variables. Together, these notions of associativity and (super)commutativity are known as “duality”, a term which arose from physics. Throughout this section we follow and extend the treatment of duality as presented in [FHL].

Let  $(V, Y(\cdot, (x, \varphi)), \mathbf{1}, \tau)$  be a vertex operator algebra with odd formal variables. Let  $V_{(n)}^*$  be the dual module of  $V_{(n)}$  for  $n \in \frac{1}{2}\mathbb{Z}$ , i.e.,  $V_{(n)}^* = \text{Hom}_{\bigwedge_*}(V, \bigwedge_*)$ . Let

$$V' = \coprod_{n \in \frac{1}{2}\mathbb{Z}} V_{(n)}^*$$

be the graded dual space of  $V$ ,

$$\bar{V} = \prod_{n \in \frac{1}{2}\mathbb{Z}} V_{(n)} = V'^*$$

the algebraic completion of  $V$ , and  $\langle \cdot, \cdot \rangle$  the natural pairing between  $V'$  and  $\bar{V}$ . We now formulate the weak supercommutativity and weak associativity properties of a vertex operator superalgebra with odd formal variables

into slightly stronger statements about “matrix coefficients” of products and iterates of vertex operators with odd formal variables.

**Proposition 9.1 (a) (rationality of products)** *For  $u, v, w \in V$ , with  $u$ , and  $v$  of homogeneous sign, and  $v' \in V'$ , the formal series*

$$\langle v', Y(u, (x_1, \varphi_1))Y(v, (x_2, \varphi_2))w \rangle,$$

*which involves only finitely many negative powers of  $x_2$  and only finitely many positive powers of  $x_1$ , lies in the image of the map  $\iota_{12}$ :*

$$\langle v', Y(u, (x_1, \varphi_1))Y(v, (x_2, \varphi_2))w \rangle = \iota_{12}f(x_1, \varphi_1, x_2, \varphi_2),$$

*where the (uniquely determined) element  $f \in \bigwedge_*[x_1, \varphi_1, x_2, \varphi_2]_{S'}$  is of the form*

$$f(x_1, \varphi_1, x_2, \varphi_2) = \frac{g(x_1, \varphi_1, x_2, \varphi_2)}{x_1^r x_2^s (x_1 - x_2 - \varphi_1 \varphi_2)^t}$$

*for some  $g \in \bigwedge_*[x_1, \varphi_1, x_2, \varphi_2]$  and  $r, s, t \in \mathbb{Z}$ .*

**(b) (supercommutativity)** *We also have*

$$\langle v', Y(v, (x_2, \varphi_2))Y(u, (x_1, \varphi_1))w \rangle = (-1)^{\eta(u)\eta(v)} \iota_{21}f(x_1, \varphi_1, x_2, \varphi_2),$$

*i.e.,*

$$\begin{aligned} \iota_{12}^{-1} \langle v', Y(u, (x_1, \varphi_1))Y(v, (x_2, \varphi_2))w \rangle = \\ (-1)^{\eta(u)\eta(v)} \iota_{21}^{-1} \langle v', Y(v, (x_2, \varphi_2))Y(u, (x_1, \varphi_1))w \rangle. \end{aligned}$$

*Proof:* Part (a) follows from the positive energy axiom (17) and truncation condition (18) for a vertex operator superalgebra. For part (b), we note that by weak supercommutativity, there exists  $k \in \mathbb{Z}_+$  such that

$$\begin{aligned} (x_1 - x_2 - \varphi_1 \varphi_2)^k \langle v', Y(u, (x_1, \varphi_1))Y(v, (x_2, \varphi_2))w \rangle = \\ (-1)^{\eta(u)\eta(v)} (x_1 - x_2 - \varphi_1 \varphi_2)^k \langle v', Y(v, (x_2, \varphi_2))Y(u, (x_1, \varphi_1))w \rangle \end{aligned} \quad (60)$$

for all  $w \in V$  and  $v' \in V'$ . From (a), we know the left-hand side of (60) involves only finitely many negative powers of  $x_2$  and only finitely many positive powers of  $x_1$ . However, the right-hand side of (60) involves only finitely many negative powers of  $x_1$  and only finitely many positive powers of  $x_2$ . Thus multiplying both sides of (60) by  $(x_1 - x_2 - \varphi_1 \varphi_2)^{-k}$  results in well-defined power series as long as on the left-hand side we expand  $(x_1 - x_2 - \varphi_1 \varphi_2)^{-k}$  in positive powers of  $x_2$  and on the right-hand side we expand  $(x_1 - x_2 - \varphi_1 \varphi_2)^{-k}$  in positive powers of  $x_1$ . The result follows.  $\square$

**Proposition 9.2 (a) (rationality of iterates)** *For  $u, v, w \in V$ , and  $v' \in V'$ , the formal series  $\langle v', Y(Y(u, (x_0, \varphi_1 - \varphi_2))v, (x_2, \varphi_2))w \rangle$ , which involves only finitely many negative powers of  $x_0$  and only finitely many positive powers of  $x_2$ , lies in the image of the map  $\iota_{20}$ :*

$$\langle v', Y(Y(u, (x_0, \varphi_1 - \varphi_2))v, (x_2, \varphi_2))w \rangle = \iota_{20}h(x_0, \varphi_1 - \varphi_2, x_2, \varphi_2),$$

where the (uniquely determined) element  $h \in \bigwedge_*[x_0, \varphi_1, x_2, \varphi_2]_{S'}$  is of the form

$$h(x_0, \varphi_1 - \varphi_2, x_2, \varphi_2) = \frac{k(x_0, \varphi_1 - \varphi_2, x_2, \varphi_2)}{x_0^r x_2^s (x_0 + x_2 - \varphi_1 \varphi_2)^t}$$

for some  $k \in \bigwedge_*[x_0, \varphi_1, x_2, \varphi_2]$  and  $r, s, t \in \mathbb{Z}$ .

**(b)** *The formal series  $\langle v', Y(u, (x_0 + x_2 + \varphi_1 \varphi_2, \varphi_1))Y(v, (x_2, \varphi_2))w \rangle$ , which involves only finitely many negative powers of  $x_2$  and only finitely many positive powers of  $x_0$ , lies in the image of  $\iota_{02}$ , and in fact*

$$\langle v', Y(u, (x_0 + x_2 + \varphi_1 \varphi_2, \varphi_1))Y(v, (x_2, \varphi_2))w \rangle = \iota_{02}h(x_0, \varphi_1 - \varphi_2, x_2, \varphi_2).$$

*Proof:* Part (a) follows from the positive energy axiom (17) and truncation condition (18) for a vertex operator superalgebra. For part (b), we note that from weak associativity, there exists  $k \in \mathbb{Z}_+$  such that

$$\begin{aligned} (x_0 + x_2 + \varphi_1 \varphi_2)^k \langle v', Y(Y(u, (x_0, \varphi_1 - \varphi_2))v, (x_2, \varphi_2))w \rangle = \\ (x_0 + x_2 + \varphi_1 \varphi_2)^k \langle v', Y(u, (x_0 + x_2 + \varphi_1 \varphi_2, \varphi_1))Y(v, (x_2, \varphi_2))w \rangle \end{aligned} \quad (61)$$

for all  $v' \in V'$ . From (a), we know the left-hand side of (61) involves only finitely many negative powers of  $x_0$  and only finitely many positive powers of  $x_2$ . However, the right-hand side of (61) involves only finitely many negative powers of  $x_2$  and only finitely many positive powers of  $x_0$ . Thus multiplying both sides of (60) by  $(x_0 + x_2 + \varphi_1 \varphi_2)^{-k}$  results in a well-defined power series as long as on the left-hand side we expand  $(x_0 + x_2 + \varphi_1 \varphi_2)^{-k}$  in positive powers of  $x_0$  and on the right-hand side we expand  $(x_0 + x_2 + \varphi_1 \varphi_2)^{-k}$  in positive powers of  $x_2$ . The result follows.  $\square$

**Proposition 9.3 (associativity)** *We have the following equality of rational functions:*

$$\begin{aligned} \iota_{12}^{-1} \langle v', Y(u, (x_1, \varphi_1))Y(v, (x_2, \varphi_2))w \rangle = \\ \left( \iota_{20}^{-1} \langle v', Y(Y(u, (x_0, \varphi_1 - \varphi_2))v, (x_2, \varphi_2))w \rangle \right) \Big|_{x_0=x_1-x_2-\varphi_1\varphi_2} \end{aligned}$$

*Proof:* Let  $f(x_1, \varphi_1, x_2, \varphi_2)$  be the rational function in Proposition 9.1. Then  $f$  satisfies

$$\iota_{02}f(x_0 + x_2 + \varphi_1\varphi_2, \varphi_1, x_2, \varphi_2) = (\iota_{12}f(x_1, \varphi_1, x_2, \varphi_2))|_{x_1=x_0+x_2+\varphi_1\varphi_2}.$$

Thus for  $h(x_0, \varphi_1 - \varphi_2, x_2, \varphi_2)$  from Proposition 9.2, we have  $h(x_0, \varphi_1 - \varphi_2, x_2, \varphi_2) = f(x_0 + x_2 + \varphi_1\varphi_2, \varphi_1, x_2, \varphi_2)$ . The result follows from Propositions 9.1 and 9.2.  $\square$

Note that rationality of products and iterates and supercommutativity and associativity imply weak supercommutativity and weak associativity. Thus by Proposition 7.3, we have:

**Proposition 9.4** *In the presence of the other axioms in the definition of vertex operator superalgebra with odd variables, the Jacobi identity follows from the rationality of products and iterates and supercommutativity and associativity. In particular, the Jacobi identity may be replaced by these properties.*

This can also be proved by using the delta-function identity (13) and the substitution rule (14).

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